WATER RESOURCES RESEARCH, VOL. 104, NO. 3, PAGES 613–621, MARCH 1999

Upscaling transmissivity under radially convergent flow in heterogeneous media

Xavier Sánchez-Vila
Departament Enginyeria del Terreny i Cartogràfica, Universitat Politècnica de Catalunya, Barcelona, Spain

Carl L. Axness
Sandia National Laboratories, Albuquerque, New Mexico, USA

Jesús Carrera
Departament Enginyeria del Terreny i Cartogràfica, Universitat Politècnica de Catalunya, Barcelona, Spain

Abstract. Most field methods used to estimate transmissivity values rely on the analysis of drawdown under convergent flow conditions. For a single well in a homogeneous and isotropic aquifer and under steady state flow conditions, drawdown s is directly related to the pumping rate Q through transmissivity T. In real, nonhomogeneous aquifers, s and Q are still directly related, now through a value called equivalent transmissivity T_{eq}. In this context, T_{eq} is defined as the value that best fits Thiem's equation and would, for example, be the transmissivity assigned to the well location in the classical interpretation of a steady state pumping test. This equivalent or upscaled transmissivity is clearly not a local value but is some representative value of a certain area surrounding the well. In this paper we present an analytical solution for upscaling transmissivities under radially convergent steady state flow conditions produced by constant pumping from a well of radius r_w in a heterogeneous aquifer based upon an extension of Thiem's equation. Using a perturbation expansion, we derive a second-order expression for T_{eq} given as a weighted average of the fluctuations in log T throughout the domain. This expression is compared to other averaging formulae from the literature, and differences are pointed out. T_{eq} depends upon an infinite series which may be expressed in terms of coefficients of the finite Fourier transform of the log transmissivity function. Sufficient conditions for convergence of this series are examined. Finally, we show that our solution agrees with existing analytical ones to second order and test the solution with a numerical example.

1. Introduction

Almost all hydrogeological books explain the relationship between drawdown and pumping rate in a homogeneous isotropic aquifer (Thiem's equation). This equation can be used to estimate transmissivity values from measured steady state drawdowns. It can also be used to provide estimations on the drawdown that are to be expected for a certain pumping rate in order to help estimate the productivity of the well as limited by the maximum drawdown allowed. These estimations are based on the analysis of drawdowns under uniform radially convergent flow conditions.

In heterogeneous media, flow toward a pumping well is still convergent but not uniformly radial. Nonuniform flow in heterogeneous media has not been addressed frequently in the literature. Early work [*Shvidler*, 1964; *Matheron*, 1967] was devoted to finding "apparent" effective transmissivity values (T_{eff}) in an annular domain with an inner and outer radii where they applied constant head boundary conditions. T_{eff} is defined in this context as the constant value of T which would provide a discharge value equal to the expectation of the discharge for the heterogeneous formation under the same boundary conditions. Apparent values range from the arithmetic mean of the T values, when the inner radius tends to zero, to the harmonic mean of the T values, when the outer radius tends to infinity. *Dagan* [1989] and *Neuman and Orr* [1993], among others, use instead a local definition for T_{eff} as the average between radial specific discharge and head gradient in an ensemble of T field realizations at a certain location. *Dagan* [1989] discusses the difference between the value of T_{eff} near the well, depending on the type of boundary conditions applied. This difference was later confirmed by different authors working either with constant flux [e.g., *Naff*, 1991] or constant head [e.g., *Indelman et al.*, 1996] boundary conditions. Recent work provided the full analytical solution for effective conductivity as a function of the radial distance to the pumping well for infinite [*Indelman and Abramovich*, 1994] and finite (annular) [*Sánchez-Vila*, 1997] domains.

The problem of upscaling transmissivity values into blocks of certain size (equivalent T values) under radially convergent mean flow has received less attention in the literature. An upscaled value is nonlocal by definition. The most important reason for upscaling is the need for incorporating measurements taken at different supports (scales of observation). A second reason is just practical: Numerical solutions involve partitioning the domains into elements that cannot be smaller than a certain size, which is generally larger than the representative scale of the measurements. While theory for upscal-

Paper number 1998WR900056.
0043-1397/99/1998WR900056$09.00

ing under parallel flow conditions is available (see *Wen and Gómez-Hernández* [1996] and *Renard and de Marsily* [1997] for recent reviews), in the radial flow case the work is mostly pseudoempirical: *Desbarats* [1992] addresses upscaling in a combined empirical-numerical approach in a two-dimensional domain, a work later expanded to three dimensions [*Desbarats*, 1994]; *Durlofsky* [1992] looks at different numerical upscaling techniques; and *Gómez-Hernández and Gorelick* [1989] use a power-averaging approach to assign block T values in a complex groundwater flow system, including several wells. On the other hand, most of the analytical work has concentrated on units composed of two bounded regions of different (fixed) T values separated by some sharp discontinuity, either radial [*Butler*, 1988] or elliptical [*Tiederman et al.*, 1995].

2. Previous Existing Averaging Formulae

In a two-dimensional domain, *Desbarats* [1992] provides a relationship for block transmissivity T_b given as a weighted spatial average of point support T values

$$T_b = \exp\left[\frac{1}{W}\int_V \frac{Y(\mathbf{x})}{r^2}\, dV\right] \tag{1}$$

where $Y(\mathbf{x}) = \ln T(\mathbf{x})$ are the point log-transmissivity values, V is the block volume, and r is the radial distance from the center of the well. In this formula, W is given by

$$W = \int_V \frac{dV}{r^2(\mathbf{x})} \tag{2}$$

The expression for T_b given in (1) is empirical, and it is based on an extension of the parallel flow case. In his paper, *Desbarats* [1992] finds (1) to work very well for multivariate log-Gaussian T fields in which the transmissivity at the well is not very different from the expected value of T taken as a random space function.

A rigorous definition of block or equivalent transmissivity T_{eq} in a two-dimensional annular domain V is the value that suits the following relationship

$$Q = 2\pi \frac{h_e - h_w}{\ln r_e/r_w} T_{\mathrm{eq}} = 2\pi A T_{\mathrm{eq}} \tag{3}$$

where Q is the pumping rate, r_w and r_e are the internal and external radii, h_w and h_e are the heads at the inner and outer radii, respectively, and $A = (h_e - h_w)/\ln (r_e/r_w)$. In short, T_{eq} is the value that best suits Thiem's formula, which is strictly valid only in confined aquifers, although under the restriction of small drawdowns can also be used in phreatic aquifers. On the basis of this definition, *Cardwell and Parsons* [1945] proved that in a heterogeneous aquifer, T_{eq} is bounded by the weighted harmonic and the weighted arithmetic averages of T over V; that is,

$$W\left[\int_V \frac{dV}{T(\mathbf{x})r^2(\mathbf{x})}\right]^{-1} \le T_{\mathrm{eq}} \le \frac{1}{W}\int_V \frac{T(\mathbf{x})\, dV}{r^2(\mathbf{x})} \tag{4}$$

Only a few simplistic cases of heterogeneity exist where an exact formula for T_{eq} can be calculated. One is the case where the transmissivity values are just a function of the radial distance to the well (e.g., when they are distributed in circular, radially symmetric annuli), which gives T_{eq} equal to the weighted harmonic mean, the lower bound in (4). In contrast, when T is constant in sectors extending from the inner to the outer radius, T_{eq} is equal to the weighted arithmetic mean, the upper bound in (4) (which in this case coincides with the simple, nonweighted, arithmetic mean). For more general cases an exact solution is not available.

In this work we develop an analytical expression for T_{eq} based on a series expansion of the flow discharge at the well in a heterogeneous domain under saturated, steady state, radially convergent mean flow conditions. The analysis can also be looked at in a different way: If we want to substitute the variable T by a single value which leads to the same drawdown (definition of T_{eq}), we need to find the relationship between the real transmissivity field (point support values) and the upscaled value. This relationship is given as a weighted average of all the values in some area surrounding the well. The upscaling formula obtained is compared with the empirical averaging formula (1) and the two bounds in (4). We will see that these four expressions all contain the same terms up to first order in the series development but differ in second-order terms.

3. Derivation of equivalent transmissivity under radially convergent, steady state flow

In the annular domain V, by setting h_w and h_e and measuring Q we use (3) to define apparent equivalent transmissivity in a heterogeneous aquifer. Our objective is to write a consistent expansion of Q and to derive the expression for the first few terms in the expansion as a function of the spatial distribution of Y.

3.1. Problem Assumptions

Before starting with the mathematical derivations we want to point out the simplifying assumptions we use and the limitations to the results brought up by them. The first assumption is that $Y_w = \ln T(r_w, \theta)$ is a constant (homogeneous T value, not depending on θ, at the well). This is not restrictive as generally the integral scale of Y is much larger than the well radius, so that the variation inside the well is negligible. Additional assumptions used in the derivations are (1) $|\langle Y\rangle - Y_w| < 1$ (where $\langle\ \rangle$ stands for expectation) and (2) $Y'(\rho, \phi)$ is continuous and of bounded variation. Assumption 1 states that the local Y_w value should not be very far from the mean, and it is used in the series expansion development. Assumption 2 is used in some of the mathematical derivations in the appendix. It should be noted that these conditions may be relaxed somewhat in the final solutions since the integrations performed tend to "smooth" out the effects of the perturbations.

A possible limitation is given by one of the boundary conditions imposed, that the hydraulic head be constant at an outer radius r_e. This is violated almost always to some degree except in the particular case of a well centered in a round island. In any case, whenever Y is highly regular (e.g., when the integral scale of Y is smaller than r_e), this condition poses no problem.

3.2. Darcy's Law

In a general domain under steady state conditions the well discharge Q can be obtained by integration of Darcy's velocity along any surface containing the well. In particular, we can use the surface of the well itself:

$$Q = -\int_0^{2\pi} q_r(r_w, \theta) r_w \, d\theta \tag{5}$$

where q_r is the radial component of Darcy's velocity (which is normal to the surface of the well at every point) integrated over the aquifer thickness. Using Darcy's law, considered valid at the local scale, we have

$$q_r(r_w, \theta) = -T_w \frac{\partial}{\partial r} h(r, \theta)|_{r=r_w} \tag{6}$$

where T_w is the transmissivity at the well, which is a constant value, as stated previously. Our next step is to compute $\partial h/\partial r|_{r=r_w}$ (that is, the head gradient at the well).

3.3. Groundwater Flow Equation

The steady state groundwater flow equation written in terms of $Y = \ln T$ is

$$\nabla^2 h + \nabla Y \nabla h = 0 \tag{7}$$

At this point we normalize the spatial coordinates by r_w. The boundary conditions in normalized polar coordinates are written as

$$\begin{aligned} h(r, \theta) &= h_w \qquad r = 1 \\ h(r, \theta) &= h_e \qquad r = R \end{aligned} \tag{8}$$

where $R = r_e/r_w$. As $Y(\mathbf{x})$ is heterogeneous, (7) cannot be fully solved analytically except for very particular cases. To overcome partially this problem, we expand the drawdown as $h = h^{(0)} + h^{(1)} + h^{(2)} + \cdots$. Approximations to h are obtained by truncating this expansion. To obtain the $h^{(i)}$ terms, we need also to expand $Y(\mathbf{x})$ as the sum of two terms: a certain mean value not depending on $\mathbf{x}$ plus a spatially variable part. As the transmissivity at the well is a very significant value, we choose its logarithm Y_w as the constant value in the decomposition; that is, $Y = Y_w + Y'$. This is an important difference from the traditional stochastic approach, where the constant value chosen is equal to the expected value of the random variable and thus the remaining term has a zero mean; in our approach, $\langle Y' \rangle \neq 0$. By substituting the expansions of h and Y into (7) we have

$$\nabla^2 h^{(0)} + \nabla^2 h^{(1)} + \nabla^2 h^{(2)} + \nabla Y' \nabla h^{(0)} + \nabla Y' \nabla h^{(1)} + \nabla Y' \nabla h^{(2)} + \cdots = 0 \tag{9}$$

(note that $\nabla Y_w = 0$). This equation can be solved in an iterative way. There are infinite possibilities for selecting the iterative procedure from this equation, although not all of them are necessarily convergent. The most convenient one is to write a set of equations so that equation i contains all the terms in $Y'^k h^{(j)}$, with $k + j = i$. These equations are then

$$\nabla^2 h^{(0)} = 0 \tag{10}$$

$$\nabla^2 h^{(i)} + \nabla Y' \nabla h^{(i-1)} = 0 \qquad i = 1, \cdots, n \tag{11}$$

This way we can find the solution for $h^{(i)}$ sequentially (starting with $h^{(0)}$). It is evident that this set of equations is equivalent to (9). There are still infinite possibilities for the expansion of h, depending on how boundary conditions are assigned. For convenience the $h^{(i)}$ functions are selected so that all the nonhomogeneous boundary conditions are applied to $h^{(0)}$

$$\begin{aligned} h^{(0)} &= h_w \qquad r = 1 \\ h^{(0)} &= h_e \qquad r = R \end{aligned} \tag{12}$$

$$h^{(i)} = 0 \tag{13}$$

at $r = 1$ and $r = R$ for $i = 1, \cdots, n$.

3.4. Evaluation of Q

Combining (5) and (6), using the expansion in h, and writing the resulting expression in the normalized coordinates, we have

$$Q = \int_0^{2\pi} T_w \left[\frac{\partial}{\partial r} h^{(0)}(r, \theta) + \frac{\partial}{\partial r} h^{(1)}(r, \theta) + \frac{\partial}{\partial r} h^{(2)}(r, \theta) + \cdots \right]_{r=1} d\theta = Q_0 + Q_1 + Q_2 + \cdots \tag{14}$$

where we are introducing the notation

$$Q_i = \int_0^{2\pi} T_w \left[\frac{\partial}{\partial r} h^{(i)}(r, \theta) \right]_{r=1} d\theta \tag{15}$$

The approach we take assumes implicitly that each of the Q_i terms is small compared to the preceeding ones so that after a few terms we are capable of capturing most of the features of Q and we can drop the higher-order terms. A necessary but not sufficient condition is that heterogeneity not be very strong (i.e., the variance of Y, $\sigma_Y^2 < 1$). Our goal is now to evaluate the first terms in the expansion of Q (up to Q_2).

3.5. Evaluation of Q_0

From (10) and (12) we can compute $h^{(0)}$ (notice that (10) is the only equation from the iterative set where Y' does not appear)

$$h^{(0)}(r) = h_w + A \ln r \tag{16}$$

Because of the radial symmetry of the domain, $h^{(0)}$ is just a function of r (normalized radial distance to the pumping well) and not of θ, and so,

$$Q_0 = 2\pi A T_w \tag{17}$$

Notice that (17) is formally equal to Thiem's formula (equation (3)) with transmissivity equal to the value at the well.

3.6. Evaluation of Q_1

From (11) and (16) we have

$$\nabla^2 h^{(1)} = -\nabla Y' \nabla h^{(0)} = -\frac{A}{r} \frac{\partial Y'}{\partial r} \tag{18}$$

with homogeneous boundary conditions. The solution to this partial differential equation can be written in terms of fundamental solutions

$$h^{(1)}(r, \theta) = A \int_V \frac{1}{\rho} \frac{\partial Y'}{\partial \rho}(\rho, \phi) G(r, \theta, \rho, \phi) \rho \, d\rho \, d\phi \tag{19}$$

where G is the Green's function corresponding to the following boundary problem:

$$\nabla^2 G(r, \theta, \rho, \phi) = -\delta(r - \rho, \theta - \phi) \tag{20}$$

$$G = 0 \qquad r = 1,\, R \tag{21}$$

The solution to this problem is given, for example, by *Weinberger* [1965]:

$$G(r,\,\theta,\,\rho,\,\phi) = \frac{\ln r \ln (R/\rho)}{2\pi \ln R} + \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2\pi n(R^n - R^{-n})} \cdot \left[\left(\frac{R}{\rho}\right)^n - \left(\frac{\rho}{R}\right)^n\right] \cos n(\theta - \phi) \tag{22}$$

valid for $\rho \geq r$ (which is our case as we will only be interested in $r = 1$, while ρ varies from 1 to R). Deriving (22) with respect to r, integrating it by parts with respect to ρ, and then inserting it into (15), we have

$$Q_1 = -T_w A \int_0^{2\pi} \int_V Y'(\rho,\,\phi)\, \frac{\partial^2 G}{\partial r \partial \rho}\, d\rho\, d\phi\, d\theta \tag{23}$$

where

$$\frac{\partial^2 G}{\partial r \partial \rho}(r,\,\theta,\,\rho,\,\phi) = -\frac{1}{2\pi\rho r \ln R} - \sum_{n=1}^{\infty} \frac{n(r^n + r^{-n})}{2\pi r \rho(R^n - R^{-n})} \cdot \left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^n\right] \cos n(\theta - \phi) \tag{24}$$

Integrating (23) over θ and taking into account that integration is carried out in $[0,\, 2\pi]$, the term in (24) including the infinite series cancels out except for $\rho = 1$, where the infinite series is not uniformly convergent and we cannot exchange the series and the integral. Nevertheless, in that case, $Y'(\rho,\,\phi) = Y'(1,\,\theta) = 0$, and so the term drops completely. Then

$$Q_1 = T_w A \int_V Y'(\rho,\,\phi)\, \frac{1}{\rho \ln R}\, d\rho\, d\phi \tag{25}$$

and from (17)

$$Q_1 = \frac{Q_0}{W} \int_V \frac{Y'(\rho,\,\phi)}{\rho^2}\, dV \tag{26}$$

where $dV = \rho\, d\rho\, d\phi$ and $W = \int_V 1/\rho^2\, dV = 2\pi \ln R$ (easily derived). This is our first important result. Notice that if the expansion of Y had been performed around a value $\bar{Y}$ different than Y_w, the discussion after (24) would not apply, and the infinite series would give a correction term which should be added to the sum of (26) plus (17) to render the first-order approximation of Q.

3.7. Evaluation of Q_2

Q_2 can be obtained similarly to Q_1, although the mathematics are much more involved. From (11)

$$\nabla^2 h^{(2)} = -\nabla Y' \nabla h^{(1)} \tag{27}$$

with homogeneous boundary conditions. The solution is given as

$$h^{(2)}(r,\,\theta) = \int_V \nabla_\rho Y' \nabla_\rho h^{(1)} G(r,\,\theta,\,\rho,\,\phi)\, dV \tag{28}$$

Then substituting $h^{(1)}$ from (19) and taking the derivative with respect to r

$$\frac{\partial h^{(2)}}{\partial r}(r,\,\theta) = \frac{\partial}{\partial r} \int_V \nabla_\rho Y' \nabla_\rho h^{(1)} G(r,\,\theta,\,\rho,\,\phi) \rho\, d\rho\, d\phi$$

$$= A \frac{\partial}{\partial r} \int_V \int_V \nabla_\rho Y'(\rho,\,\phi) \frac{\partial Y'}{\partial \rho^*}(\rho^*,\,\phi^*) \cdot \nabla_\rho G(\rho,\,\phi,\,\rho^*,\,\phi^*) G(r,\,\theta,\,\rho,\,\phi) \frac{1}{\rho^*}\, dV\, dV^* \tag{29}$$

with $dV^* = \rho^*\, d\rho^*\, d\phi^*$; after very involved manipulation we find that Q_2 can be written as the sum of three terms, which we call Q_{2a}, Q_{2b}, and Q_{2c}, equal to (see the appendix for the derivations)

$$Q_2 = Q_{2a} + Q_{2b} + Q_{2c}$$

$$= \frac{-Q_0}{2W} \int_V \frac{Y'^2(\rho,\,\phi)}{\rho^2}\, dV + \frac{Q_0}{W^2} \left[\int_V \frac{Y'(\rho,\,\phi)}{\rho^2}\, dV\right]^2 + \frac{Q_0}{W} \sum_{n=1}^{\infty} \int_V \int_V \frac{Y'(\rho,\,\phi) Y'(\rho^*,\,\phi^*)}{\rho^2 \rho^{*2}} \cdot H_n(\rho,\,\phi,\,\rho^*,\,\phi^*)\, dV\, dV^* \tag{30}$$

where

$$H_n = \begin{cases} \dfrac{n}{2\pi}\left(\dfrac{\rho^n + \rho^{-n}}{R^n - R^{-n}}\right)\left[\left(\dfrac{R}{\rho^*}\right)^n + \left(\dfrac{\rho^*}{R}\right)^n\right] \cos n(\phi - \phi^*) & (\rho^* > \rho) \\ \dfrac{n}{2\pi}\left(\dfrac{\rho^{*n} + \rho^{*-n}}{R^n - R^{-n}}\right)\left[\left(\dfrac{R}{\rho}\right)^n + \left(\dfrac{\rho}{R}\right)^n\right] \cos n(\phi - \phi^*) & (\rho^* \leq \rho) \end{cases} \tag{31}$$

Note that $Q_{2b} = Q_1^2/Q_0$. Finally, using (3) and substituting Q by its approximation up to second order, we have

$$T_{eq} = \frac{1}{2\pi A} (Q_0 + Q_1 + Q_{2a} + Q_{2b} + Q_{2c}) \tag{32}$$

4. Analysis of the Solution

Using the notation already introduced, it is possible to write the second-order expansions of both the formula by *Desbarats* [1992],

$$T_{eq} = \frac{1}{2\pi A}\left(Q_0 + Q_1 + \frac{1}{2} Q_{2b}\right) \tag{33}$$

that corresponding to the weighted harmonic mean,

$$T_{eq} = \frac{1}{2\pi A} (Q_0 + Q_1 + Q_{2a} + Q_{2b}) \tag{34}$$

and that corresponding to the weighted arithmetic mean,

$$T_{eq} = \frac{1}{2\pi A} (Q_0 + Q_1 - Q_{2a}) \tag{35}$$

In order to get (33)–(35) we have expanded the corresponding formulae around T_w so they can be directly compared with (32). These same formulae can be obtained in a more compact form from (11) of *Desbarats* [1994] by expanding his expressions around T_w. Using our notation and expanding the exponential up to second order, (11) [*Desbarats*, 1994] reads:

$$T_{eq} = \exp(Y_w)\left[1 + Y'_v + \frac{1}{2}(Y'_v)^2 + \frac{w}{2W}\int_V \frac{(Y' - Y'_v)^2}{r^2}\,dV\right] \tag{36}$$

with

$$Y'_v = \frac{1}{W}\int_V \frac{Y'}{r^2}\,dV \tag{37}$$

where w is the power-weighting value. From this expression we can finally write a general formula for the approximation of the equivalent transmissivity which includes (33)–(35)

$$T_{eq} = \frac{1}{2\pi A}\left[Q_0 + Q_1 + \frac{1}{2}Q_{2b} + w\left(-Q_{2a} - \frac{1}{2}Q_{2b}\right)\right] \tag{38}$$

Now, taking $w = 0$, (38) is equivalent to Desbarats' formula, while for $w = -1$ and $w = +1$ the resulting expressions are equal to the weighted harmonic (equation (34)) and arithmetic (equation (35)) means.

It is immediately apparent from the above formulae that our second-order solution (equation (32)) agrees with the weighted harmonic mean (34) when Q_{2c} is equal to zero. That is the case, for example, when the transmissivity field does not depend on θ since the cosine term is integrated over its full range. Thus, we have a simple verification of (32) in one limiting case.

We note that up to first order in the expansion there are no differences in the terms for any of the previous formulae; however, differences in the second-order terms do appear. The importance of the second-order term will be field dependent. In multi-Gaussian random ln T fields, *Desbarats* [1992] shows numerically that (33) gives an excellent result. In non-multi-Gaussian fields this need not be the case. We will see in the next section a synthetic example where the term Q_{2c} is important and needs to be considered. In real cases the importance of the Q_{2c} term should be established before using any of the simplified formulae for upscaling purposes.

5. Synthetic Example

We have shown in the previous section that in a case where the transmissivity is only a function of the radius our expression is exact to second order. However, this cannot be considered a real example. There is still some discussion in the scientific community about whether multi-Gaussianity is a proper reflection of real heterogeneity. In this section we want to point out that this assumption is critical for upscaling under radial conditions. While *Desbarats'* [1994] formula is applicable to multi-Gaussian fields, we want to show with a synthetic example that in non-multi-Gaussian fields there is a need for more complicated formulae, such as the one presented in this paper.

For that purpose we consider a test field with a known space function $Y' = \beta\cos(m\theta)$. Here β is a real parameter, and m is an integer >1. Around the well we fix an annulus with constant $Y' = 0$. This annulus extends from the well ($r = 1$) to $r = 1.5$. The variable Y' function extends from there to the outer radius. The field is represented in Figure 1.

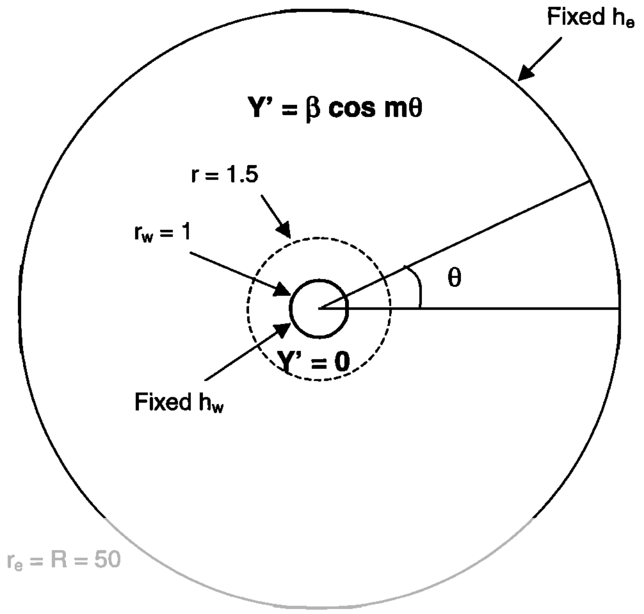

Figure 1. Geometry of the log transmissivity Y field used in the synthetic example; $Y_w = 0$, and so, $Y = Y'$.

For this particular T field the integrals appearing in the Q_i terms can be calculated analytically in terms of Q_0: $Q_1 = 0$, $Q_{2a} = -Q_0\beta^2/4$, $Q_{2b} = 0$, and $Q_{2c} = Q_0\beta^2/2$. As a result, $Q_2/Q_0 = \beta^2/4$. The comparison between the numerically calculated Q and the analytical Q_0 and Q_2 as a function of the parameter β is shown in Figure 2 for the case $m = 3$. Figure 2 clearly shows a range, $\beta \le 1$, in which the numerically calculated Q is approximately constant and well represented by Q_0 (note that in this particular case, Q_0 corresponds to (33)), while (35) gives the exact solution). When $1 \le \beta \le 10$, the flow at the well is dominated by the high T regions, and the addition of Q_2 is necessary for the perturbed solution to accurately reproduce the numerical solution. The flat region for $\beta \ge 10$

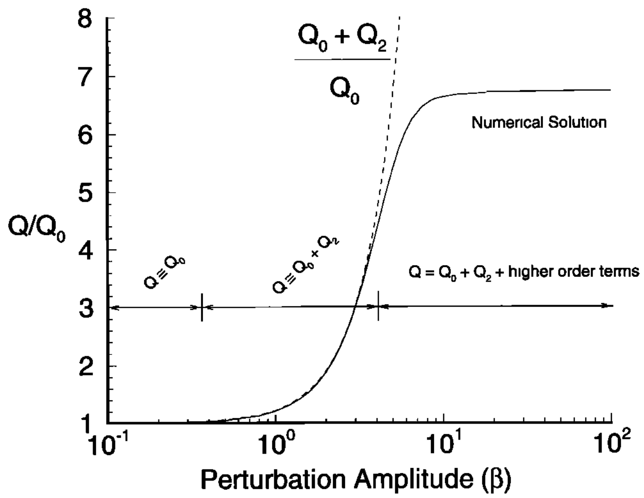

Figure 2. Normalized Q versus perturbation amplitude in the analytical and numerical solutions for the synthetic example.

corresponds to the range over which the flow to the well is controlled by the area near the wellbore where the transmissivity is small compared to the amplitude of the cosine function. It is apparent that for $\beta \geq 4$, terms higher than second order are necessary to accurately calculate Q. Notice that as the variance can be calculated from

$$\sigma_Y^2 = \frac{1}{2\pi} \int_0^{2\pi} \beta^2 \cos^2 (m\theta)\, d\theta = \frac{\beta^2}{2} \tag{39}$$

the case $\beta = 4$ corresponds to a very high variance ($\sigma_Y^2 = 8$), well above the traditional limit $\sigma_Y^2 < 1$.

6. Expansion of $Y'(\rho, \theta)$ in Terms of the Finite Fourier Series

It is apparent from our analysis that successive integration by parts on the Green's function integral are done possibly at the expense of convergence of the associated infinite series term. It is not obvious from the form of (30) under what conditions on $Y'(\rho, \theta)$ the infinite sum converges. In this section we develop the relationship between the coefficients of the finite Fourier series representation of $Y'(\rho, \theta)$ and the subsequent Q_i and T_{eq} terms. We then find sufficient conditions under which the sum of (30) converges in terms of the Fourier coefficients. We begin by assuming a finite Fourier representation for $Y'(\rho, \theta)$ [see *Weinberger*, 1965].

$$Y'(\rho, \theta) = a_0(\rho)/2 + \sum_{m=1}^{\infty} a_m(\rho) \cos (m\theta) + b_m(\rho) \sin (m\theta) \tag{40}$$

Substituting into (26) and (30) gives the relationships

$$Q_1 = Q_0 c_{01}/2 \tag{41}$$

$$Q_2 = Q_0 \left\{ -\frac{c_{02}}{8} - \frac{1}{4} \sum_{m=1}^{\infty} (c_{m2} + d_{m2}) + \frac{c_{01}^2}{4} + \frac{\pi}{2 \ln R} \sum_{m=1}^{\infty} \int_1^R \cdot \int_1^R \frac{H_m(\rho, \rho')}{\rho\rho'} [a_m(\rho)a_m(\rho') + b_m(\rho)b_m(\rho')]\, d\rho\, d\rho' \right\} \tag{42}$$

where $H_m(\rho, \rho') = H_m(\rho, \rho', 0, 0)$, $c_{ij} = [\int_1^R a_i^j(\rho)\, d\rho/\rho]/\ln R$, and $d_{ij} = [\int_1^R b_i^j(\rho)\, d\rho/\rho]/\ln R$. From the above relations, T_{eq} becomes

$$T_{\text{eq}} = T_w \left\{ 1 + \frac{c_{01}}{2} - \frac{c_{02}}{8} + \frac{c_{01}^2}{4} - \frac{1}{4} \sum_{m=1}^{\infty} (c_{m2} + d_{m2}) + \frac{\pi}{2 \ln R} \sum_{m=1}^{\infty} \cdot \int_1^R \int_1^R \frac{H_m(\rho, \rho')}{\rho\rho'} [a_m(\rho)a_m(\rho') + b_m(\rho)b_m(\rho')]\, d\rho\, d\rho' \right\} \tag{43}$$

In order to obtain these solutions we have used the orthogonality of the sine and cosine functions. Note that the integration of the last term in (43) is considerably simpler than that given in (30).

Expressing Y' in terms of a Fourier series permits us to use the extensive knowledge about Fourier series to increase our understanding of the properties of the solution. In particular, we can examine the convergence properties of the sum of the coefficients and draw conclusions about the necessary conditions upon $Y'(\rho, \theta)$ for convergence of the sum. We state our terms of convergence in the form of a theorem.

Let S_{H_n} be the final sum of (30) and suppose that $Y'(\rho, \theta)$ is expressible as a finite Fourier series. Suppose further that the Fourier coefficients are bounded as

$$|a_m(\rho)|,\ |b_m(\rho)| \leq \frac{f(\rho)}{m^a} \tag{44}$$

where $f(\rho)$ is a monotonically increasing function independent of m and $a > 1/2$. Then S_{H_n} is absolutely convergent and has an upper bound

$$|S_{H_n}| \leq f^2(R)\zeta(a) \tag{45}$$

where $\zeta(a)$ is the Riemann Zeta function. As a particular case, the sum converges when $Y'(\rho, \theta)$ is of bounded variation (meaning that for any partition P_i of the interval $[0, 2\pi]$ the sum of the absolute differences of $|\nabla Y'(\rho, \theta_i)|$ are bounded), in which case, $|a_m(\rho)|$, $|b_m(\rho)| \sim O(1/m)$.

In order to prove the theorem we use (31) to get an expression corresponding to the final sum in terms of just the $a_m(\rho)$ terms:

$$S_{H_{n,a}} = \frac{\pi}{2 \ln R} \sum_{m=1}^{\infty} \int_1^R \int_1^R \frac{H_m(\rho, \rho')}{\rho\rho'} a_m(\rho)a_m(\rho')\, d\rho\, d\rho'$$

$$= \frac{\pi}{2 \ln R} \sum_{m=1}^{\infty} \frac{m}{2\pi(R^m - R^{-m})} \int_1^R \frac{a_m(\rho)}{\rho} \left\{ \left[\left(\frac{R}{\rho}\right)^m + \left(\frac{R}{\rho}\right)^{-m} \right] \int_1^\rho \frac{a_m(\rho')}{\rho'} (\rho'^m + \rho'^{-m})\, d\rho' + (\rho^m + \rho^{-m}) \int_\rho^R \frac{a_m(\rho')}{\rho'} \left[\left(\frac{R}{\rho'}\right)^m + \left(\frac{R}{\rho'}\right)^{-m} \right] d\rho' \right\} d\rho \tag{46}$$

A similar term may be written for the $b_m(\rho)$ term. Using the triangle inequality and because both $a_m(\rho)$ and $b_m(\rho)$ are bounded by (44), we may bound the entire sum as

$$|S_{H_n}| \leq \frac{f^2(R)}{2 \ln R} \sum_{m=1}^{\infty} \frac{m^{-2a}}{(R^m - R^{-m})} \int_1^R \frac{1}{\rho} \left\{ \left[\left(\frac{R}{\rho}\right)^m + \left(\frac{R}{\rho}\right)^{-m} \right] \cdot (\rho^m - \rho^{-m}) - (\rho^m - \rho^{-m}) \left[\left(\frac{R}{\rho}\right)^m - \left(\frac{R}{\rho}\right)^{-m} \right] d\rho' \right\} \cdot d\rho = f^2(R) \sum_{m=1}^{\infty} \frac{1}{m^{2a}} = f^2(R)\zeta(a) \tag{47}$$

where we have used the monotonicity of f (so that $f(\rho) \leq f(R)$). This proves our theorem.

Note that the condition $a > 1/2$ is necessary for convergence of the final sum (the Riemann Zeta function). This condition insures absolute convergence of S_{H_n}, but nothing can be said about the convergence of S_{H_n} when $a \leq 1/2$. Under this condition the Riemann Zeta function diverges, but our earlier use of the triangle inequality might conceivably have made $|S_{H_n}|$ divergent while making S_{H_n} convergent (conditional convergence). Finally, if we assume for any particular value of ρ that $Y'(\rho, \theta)$ is of bounded variation with respect to θ, then it is possible to prove [*Apostol*, 1975] that each of the Fourier coefficients is of order $O(1/m)$. This case easily meets our convergence criterion.

7. Conclusions

As it is well known, transmissivity derived from the interpretation of a pumping test is not only dependent upon local conditions in the area of the well, but it represents an average value over a representative volume of aquifer. In this paper we derive an analytical expression for the weighting function involved in the averaging process, which is valid independently of the univariate or multivariate distributions of the T field. In order to get this weighting function we perform a second-order expansion for the flow to a well in a two-dimensional heterogeneous medium assuming constant head boundary conditions at the wellbore and an external radius. The perturbed solution is similar to a generalized form of the *Craft and Hawkins* [1959] solution but contains a term dependent upon the Green's function of the domain. This term may be very important under preferential flow conditions.

The solution is verified by comparison with the exact solution for a radial case and with numerical simulations using a periodic Y function. Ranges in which the zero-, second-, and higher-order terms of the perturbed solution are necessary for accurate calculation of the numerical solution were observed in this example. We further show that the results for simplified cases agree with common knowledge, and we give some conditions for the transmissivity function under which our averaging solution, which is expressed in an integral form, converges.

This work poses a note of caution when using simplified formulae for upscaling purposes under radial flow conditions. While these formulae have been shown to work in T fields that display some regularity (e.g., log-T being multi-Gaussian), in other types of fields more complicated formulae, such as the one presented in this paper, might be necessary.

Appendix: Derivation of the Q_2 Term

The definition of Q_2 is given by (15):

$$Q_2 = T_w \int_0^{2\pi} \left[\frac{\partial}{\partial r} h^2(r, \theta) \right]_{r=1} d\theta \tag{A1}$$

From (29) we have

$$Q_2 = AT_w \int_0^{2\pi} \int_V \int_V \nabla_\rho Y'(\rho, \phi) \frac{\partial Y'}{\partial \rho^*}(\rho^*, \phi^*) \nabla_\rho G(\boldsymbol{\rho}, \boldsymbol{\rho}^*) \cdot \left[\frac{\partial}{\partial r} G(r, \theta, \rho, \phi) \right]_{r=1} \frac{1}{\rho^*} dV\, dV^*\, d\theta \tag{A2}$$

Integrating with respect to θ,

$$Q_2 = AT_w \int_V \int_V \nabla_\rho Y'(\rho, \phi) \frac{\partial Y'}{\partial \rho^*}(\rho^*, \phi^*) \cdot \nabla_\rho G(\boldsymbol{\rho}, \boldsymbol{\rho}^*) \frac{\ln (R/\rho)}{\ln R} \frac{1}{\rho^*} dV\, dV^* \tag{A3}$$

Let $U = Y'(\rho, \phi)$ and $V = \nabla_\rho G\ (\ln R/\rho)/(\ln R)$, then applying Green's Formula,

$$Q_2 = AT_w \left\{ \int_V \frac{\partial Y'}{\partial \rho^*}(\rho^*, \phi^*) \left[\int_S Y'(\rho, \phi) \cdot \frac{\ln R/\rho}{\ln R} \nabla_\rho G(\boldsymbol{\rho}, \boldsymbol{\rho}^*) \cdot d\mathbf{S} - \int_V Y'(\rho, \phi) \cdot \nabla_\rho \left(\frac{\ln (R/\rho)}{\ln R} \nabla \rho G(\boldsymbol{\rho}, \boldsymbol{\rho}^*) \right) dV \right] d\rho^*\, d\phi^* \right\} \tag{A4}$$

where a surface integral appears. As for the inner boundary $Y'(\rho = 1, \phi) = 0$ and for the outer one $\ln (R/\rho) = 0$, we have

$$\int_S Y'(\rho, \phi) \frac{\ln R/\rho}{\ln R} \nabla_\rho G(\boldsymbol{\rho}, \boldsymbol{\rho}^*) \cdot d\mathbf{S} = 0 \tag{A5}$$

After cancelling this integral we can decompose Q_2 into two terms: $Q_2 = Q_\alpha + Q_\beta$, where

$$Q_\alpha = -AT_w \int_V \int_V \frac{\partial Y'}{\partial \rho^*}(\rho^*, \phi^*) Y'(\rho, \phi) \cdot \nabla_\rho \frac{\ln (R/\rho)}{\ln R} \nabla_\rho G(\boldsymbol{\rho}, \boldsymbol{\rho}^*)\, dV\, d\rho^*\, d\phi^* \tag{A6}$$

$$Q_\beta = AT_w \int_V \int_V \frac{\partial Y'}{\partial \rho^*}(\rho^*, \phi^*) Y'(\rho, \phi) \cdot \frac{1}{\rho \ln R} \frac{\partial}{\partial \rho} G(\boldsymbol{\rho}, \boldsymbol{\rho}^*)\, dV\, d\rho^*\, d\phi^* \tag{A7}$$

Noting that $\nabla_\rho^2 G(\rho, \phi, \rho^*, \phi^*) = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}^*)$,

$$Q_\alpha = \frac{AT_w}{\ln R} \int_0^{2\pi} \int_1^R \frac{\partial Y'}{\partial \rho}(\rho, \phi) Y'(\rho, \phi) \ln (R/\rho)\, d\rho\, d\phi \tag{A8}$$

and as $Y' \partial Y'/\partial \rho = 1/2 \partial Y'^2/\partial \rho$, integrating by parts with respect to ρ,

$$Q_\alpha = \frac{AT_w}{\ln R} \left\{ \int_0^{2\pi} \left(\frac{Y'^2}{2} \ln (R/\rho) \right) \Bigg|_1^R d\phi + \int_0^{2\pi} \int_1^R \frac{Y'^2(\rho, \phi)}{2\rho} d\rho\, d\phi \right\} = \frac{\pi AT_w}{W} \int_V \frac{Y'^2(\rho, \phi)}{\rho^2} dV \tag{A9}$$

with $W = 2\pi \ln R$ (already defined in the text). Q_β can be written from (A7) after integration by parts in ρ^* as

$$Q_\beta = \frac{AT_w}{\ln R} \int_V \int_0^{2\pi} Y'(\rho^*, \phi^*) Y'(\rho, \phi) \frac{1}{\rho} \frac{\partial G}{\partial \rho} (\boldsymbol{\rho}, \boldsymbol{\rho}^*/1 \le \rho^* \le \rho)|_1^\rho \, dV \, d\phi^* + \frac{AT_w}{\ln R} \int_V \int_0^{2\pi} Y'(\rho^*, \phi^*) Y'(\rho, \phi)$$

$$\cdot \frac{1}{\rho} \frac{\partial G}{\partial \rho} (\boldsymbol{\rho}, \boldsymbol{\rho}^*/\rho \le \rho^* \le R)|_\rho^R \, dV \, d\phi^* - \frac{AT_w}{\ln R} \int_V \int_0^{2\pi} \int_1^\rho$$

$$\cdot Y'(\rho^*, \phi^*) Y'(\rho, \phi) \frac{1}{\rho} \frac{\partial}{\partial \rho^*} \left[\frac{\partial G}{\partial \rho} (\boldsymbol{\rho}, \boldsymbol{\rho}^*) \right] dV \, d\rho^*$$

$$\cdot d\phi^* - \frac{AT_w}{\ln R} \int_V \int_0^{2\pi} \int_\rho^R Y'(\rho^*, \phi^*) Y'(\rho, \phi)$$

$$\cdot \frac{1}{\rho} \frac{\partial}{\partial \rho^*} \left[\frac{\partial G}{\partial \rho} (\boldsymbol{\rho}, \boldsymbol{\rho}^*) \right] dV \, d\rho^* \, d\phi^* \qquad \text{(A10)}$$

Note that the discontinuity in $\partial G/\partial \rho$ is handled by considering the integral over the intervals $[0, \rho)$ and $(\rho, R]$, where it is continuous assuming Y' to be continuous. An alternate approach would be to write $\partial G/\partial \rho$ as a generalized function and integrate over the entire range. In this case the delta function introduced by the second derivative of G would take care of the contributions of the discontinuities. The derivative of G is given by the sum of two terms: $\partial G/\partial \rho = G_{\rho 1} + G_{\rho 2}$ where

$$G_{\rho 1} = \begin{cases} \dfrac{\ln (R/\rho^*)}{2\pi\rho \ln R} & (\rho < \rho^*) \\ \dfrac{-\ln \rho^*}{2\pi\rho \ln R} & (\rho^* < \rho) \end{cases} \qquad \text{(A11)}$$

while the $G_{\rho 2}$ term includes an infinite series

$$G_{\rho 2} = \begin{cases} \dfrac{1}{2\pi\rho} \displaystyle\sum_{n=1}^{\infty} \frac{\rho^n + \rho^{-n}}{R^n - R^{-n}} \left[\left(\frac{R}{\rho^*}\right)^n - \left(\frac{\rho^*}{R}\right)^n \right] \cos n(\phi - \phi^*) & (\rho < \rho^*) \\ \dfrac{-1}{2\pi\rho} \displaystyle\sum_{n=1}^{\infty} \frac{\rho^{*n} - \rho^{*-n}}{R^n - R^{-n}} \left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^n \right] \cos n(\phi - \phi^*) & (\rho^* < \rho) \end{cases}$$

$$\text{(A12)}$$

Now $Q_{\beta 1}$ will be the contribution to Q_β given by $G_{\rho 1}$. Substituting (A11) into (A10) and rearranging terms gives

$$Q_{\beta 1} = AT_w \left\{ \frac{-1}{W} \int_0^{2\pi} \int_V \frac{Y'(\rho, \phi) Y'(\rho, \phi^*)}{\rho^2} dV \, d\phi^* + \frac{2\pi}{W^2} \int_V \int_V \frac{Y'(\rho, \phi)}{\rho^2} \frac{Y'(\rho^*, \phi^*)}{\rho^{*2}} dV \, dV^* \right\} \qquad \text{(A13)}$$

On the other hand, we term $Q_{\beta 2}$ the contribution to Q_β given by $G_{\rho 2}$. In order to evaluate this term we define the function

$$S_N = \begin{cases} \dfrac{1}{2\pi\rho} \displaystyle\sum_{n=1}^{N} \frac{\rho^n + \rho^{-n}}{R^n - R^{-n}} \left[\left(\frac{R}{\rho^*}\right)^n - \left(\frac{\rho^*}{R}\right)^n \right] \cos n(\phi - \phi^*) & (\rho < \rho^*) \\ \dfrac{-1}{2\pi\rho} \displaystyle\sum_{n=1}^{N} \frac{\rho^{*n} - \rho^{*-n}}{R^n - R^{-n}} \left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^n \right] \cos n(\phi - \phi^*) & (\rho^* < \rho) \end{cases}$$

$$\text{(A14)}$$

so that $\lim_{N\to\infty} S_N = G_{\rho 2}(\rho, \phi, \rho^*, \phi^*)$. Now, assuming $Q_{\beta 2}$ exists, it is equal to

$$Q_{\beta 2} = \lim_{N\to\infty} \frac{AT_w}{\ln R} \int_V \int_0^{2\pi} Y'(\rho, \phi^*) Y'(\rho, \phi) \frac{1}{\rho} S_N|_1^\rho \, d\phi^* \, dV$$

$$+ \lim_{N\to\infty} \frac{AT_w}{\ln R} \int_V \int_0^{2\pi} Y'(\rho, \phi^*) Y'(\rho, \phi) \frac{1}{\rho} S_N|_\rho^R \, d\phi^* \, dV$$

$$- \lim_{N\to\infty} \frac{AT_w}{\ln R} \int_V \int_V Y'(\rho^*, \phi^*) Y'(\rho, \phi) \frac{1}{\rho} \frac{\partial S_N}{\partial \rho^*} d\rho^* \, d\phi^* \, dV$$

$$\text{(A15)}$$

The contribution to $Q_{\beta 2}$ of the first two terms in (A15) is termed $Q_{\beta 2,a}$, while we call the third term $Q_{\beta 2,b}$. Then

$$Q_{\beta 2,a} = -\frac{AT_w}{\ln R} \int_V \frac{1}{\rho} Y'(\rho, \phi) \cdot \left[\lim_{N\to\infty} \int_0^{2\pi} Y'(\rho, \phi^*) \frac{1}{\pi\rho} \sum_{n=1}^{N} \cos n(\phi - \phi^*) \, d\phi^* \right] dV$$

$$= -\frac{AT_w}{\ln R} \int_V \frac{1}{\rho^2} Y'(\rho, \phi) \lim_{N\to\infty} I_N \, dV \qquad \text{(A16)}$$

where I_N can be written in terms of $D_N(\phi - \phi^*)$, which is Dirichlet's kernel [*Apostol*, 1975, pp. 317–319].

$$I_N = \int_0^{2\pi} \frac{1}{\pi} Y'(\rho, \phi^*) \left[D_N(\phi - \phi^*) - \frac{1}{2} \right] d\phi^* \qquad \text{(A17)}$$

Following Apostol's arguments, if $Y'(\rho, \phi^*)$ is of bounded variation on the compact interval $[\phi - \delta, \phi + \delta]$ for some $\delta < \pi$, then the limit of I_N exists and is equal to

$$\lim_{N\to\infty} I_N = \left[\lim_{t\to\infty} \frac{Y'(\rho, \phi + t) + Y'(\rho, \phi - t)}{2} - \frac{1}{2\pi} \int_0^{2\pi} Y'(\rho, \phi^*) \, d\phi^* \right] \qquad \text{(A18)}$$

If we further assume that $Y'(\rho, \phi)$ is continuous, then

$$\lim_{N\to\infty} I_N = Y'(\rho, \phi) - \frac{1}{2\pi} \int_0^{2\pi} Y'(\rho, \phi^*) \, d\phi^* \qquad \text{(A19)}$$

so that, finally, $Q_{\beta 2,a}$ becomes

$$Q_{\beta 2,a} = \frac{AT_w}{\ln R}\left\{ -\int_V \frac{Y'^2(\rho, \phi)}{\rho^2}\, dV + \frac{1}{2\pi}\int_0^{2\pi}\int_V \cdot \frac{Y'(\rho, \phi)Y'(\rho, \phi^*)}{\rho^2}\, dV\, d\phi^* \right\} \tag{A20}$$

On the other hand, $Q_{\beta 2,b}$ is equal to

$$Q_{\beta 2,b} = \frac{AT_w}{\ln R}\sum_{n=1}^{\infty}\int_V\int_V \frac{Y'(\rho^*, \phi^*)}{\rho^*}\frac{Y'(\rho, \phi)}{\rho^2} \cdot H_n(\rho, \phi, \rho^*, \phi^*)\, d\rho^*\, d\phi^*\, dV \tag{A21}$$

where

$$H_n = \begin{cases} \dfrac{n}{2\pi}\left(\dfrac{\rho^n + \rho^{-n}}{R^n - R^{-n}}\right)\left[\left(\dfrac{R}{\rho^*}\right)^n + \left(\dfrac{\rho^*}{R}\right)^n\right] & \cos n(\phi - \phi^*) \quad (\rho^* > \rho) \\ \dfrac{n}{2\pi}\left(\dfrac{\rho^{*n} + \rho^{*-n}}{R^n - R^{-n}}\right)\left[\left(\dfrac{R}{\rho}\right)^n + \left(\dfrac{\rho}{R}\right)^n\right] & \cos n(\phi - \phi^*) \quad (\rho^* < \rho) \end{cases} \tag{A22}$$

Adding all the terms in (A13), (A20), and (A21), it can be seen that one of the terms in $Q_{\beta 2,a}$ cancels out one in $Q_{\beta 1}$; finally, we can write Q_β as

$$Q_\beta = \frac{2\pi AT_w}{W}\left\{ -\int_V \frac{Y'^2(\rho, \phi)}{\rho^2}\, dV + \frac{1}{W}\int_V\int_V \cdot \frac{Y'(\rho, \phi)}{\rho^2}\frac{Y'(\rho^*, \phi^*)}{\rho^{*2}}\, dV\, dV^* + \sum_{n=1}^{\infty}\int_V\int_V \cdot \frac{Y'(\rho, \phi)}{\rho^2}\frac{Y'(\rho^*, \phi^*)}{\rho^{*2}} H_n(\rho, \phi, \rho^*, \phi^*)\, dV\, dV^* \right\} \tag{A23}$$

Note that the first term is -2 times Q_α. Thus Q_2, given as the sum of (A9) plus (A23), may be finally written as the sum of three terms:

$$Q_2 = \frac{2\pi AT_w}{W}\left\{ \frac{-1}{2}\int_V \frac{Y'^2(\rho, \phi)}{\rho^2}\, dV + \frac{1}{W}\left[\int_V \frac{Y'(\rho, \phi)}{\rho^2}\, dV\right]^2 + \sum_{n=1}^{\infty}\int_V\int_V \frac{Y'(\rho, \phi)Y'(\rho^*, \phi^*)}{\rho^2\rho^{*2}} \cdot H_n(\rho, \phi, \rho^*, \phi^*)\, dV\, dV^* \right\} \tag{A24}$$

which corresponds exactly to (30).

Acknowledgments. This work was funded by Spanish Nuclear Waste Management Company (ENRESA) and the EU. The second author is grateful for the support provided by the Sandia National Laboratories DSP Program. The authors want to thank A. Desbarats, J. J. Gómez-Hernández, and an anonymous scientist for their thorough reviews.

References

Apostol, T. M., *Mathematical Analysis*, 483 pp., Addison-Wesley, Reading, Mass., 1975.

Butler, J. J., Jr., Pumping tests in nonuniform aquifers: The radially symmetric case, *J. Hydrol.*, *101*, 15–30, 1988.

Cardwell, W. T., and R. L. Parsons, Average permeabilities of heterogeneous oil sands, *Trans. Am. Inst. Min. Metall. Pet. Eng.*, *160*, 34–42, 1945.

Craft, B. C., and M. F. Hawkins, *Applied Petroleum Reservoir Engineering*, 437 pp., Prentice-Hall, Englewood Cliffs, N. J., 1959.

Dagan, G., *Flow and Transport in Porous Formations*, 465 pp., Springer-Verlag, New York, 1989.

Desbarats, A. J., Spatial averaging of transmissivity in heterogeneous fields with flow toward a well, *Water Resour. Res.*, *28*(3), 757–767, 1992.

Desbarats, A. J., Spatial averaging of hydraulic conductivity under radial flow conditions, *Math. Geol.*, *26*(1), 1–21, 1994.

Durlofsky, L. J., Representation of grid block permeability in coarse scale models of randomly heterogeneous porous media, *Water Resour. Res.*, *28*(7), 1791–1800, 1992.

Gómez-Hernández, J. J., and S. M. Gorelick, Effective groundwater model parameter values: Influence of spatial variability of hydraulic conductivity, leakance and recharge, *Water Resour. Res.*, *25*(4), 405–419, 1989.

Indelman, P., and B. Abramovich, Nonlocal properties of nonuniform averaged flows in heterogeneous media, *Water Resour. Res.*, *30*(12), 3385–3393, 1994.

Indelman, P., A. Fiori, and G. Dagan, Steady flow toward wells in heterogeneous formations: Mean head and equivalent conductivity, *Water Resour. Res.*, *32*(7), 1975–1983, 1996.

Matheron, G., *Elements Pour une Theorie des Milieux Poreux*, 166 pp., Masson et Cie, Paris, 1967.

Naff, R. L., Radial flow in heterogeneous porous media: An analysis of specific discharge, *Water Resour. Res.*, *27*(3), 307–316, 1991.

Neuman, S. P., and S. Orr, Prediction of steady state flow in nonuniform geologic media by conditional moments: Exact nonlocal formalism, effective conductivities, and weak approximation, *Water Resour. Res.*, *29*(2), 341–364, 1993.

Renard, P., and G. de Marsily, Calculating equivalent permeability: A review, *Adv. Water Resour.*, *20*(5–6), 253–278, 1997.

Sánchez-Vila, X., Radially convergent flow in heterogeneous porous media, *Water Resour. Res.*, *33*(7), 1633–1641, 1997.

Shvidler, M. I., *Filtration Flows in Heterogeneous Media (a Statistical Approach)*, Consult. Bur., New York, 1964.

Tiederman, C. R., P. A. Hsieh, and S. B. Christian, Characterization of a high-transmissivity zone by well test analysis: Steady state case, *Water Resour. Res.*, *31*(1), 27–37, 1995.

Weinberger, H. F., *Partial Differential Equations*, Blaisdell, Waltham, Mass., 1965.

Wen, X.-H., and J. J. Gómez-Hernández, Upscaling hydraulic conductivities in heterogeneous media: An overview, *J. Hydrol.*, *183*, ix–xxxii, 1996.

C. L. Axness, Sandia National Laboratories, Albuquerque, NM 87185.

J. Carrera and X. Sánchez-Vila, Departamento Enginyeria del Terreny i Cartogràfica, Universitat Politècnica de Catalunya, 08034 Barcelona, Spain. (dsanchez@etseccpb.upc.es)

(Received January 14, 1998; revised October 13, 1998; accepted October 13, 1998.)